AP Physics C 2006: Free Response Question 2

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Given the circuit shown in Figure 1, we wish to determine the currents I_1 and I_2 as a function of time. We are given the numerical values: $\mathcal{E} = 12$ V, $C = 0.060$ F, and $R_1 = R_2 = 4700\ \Omega$, although they are not really relevant to this part of the problem.

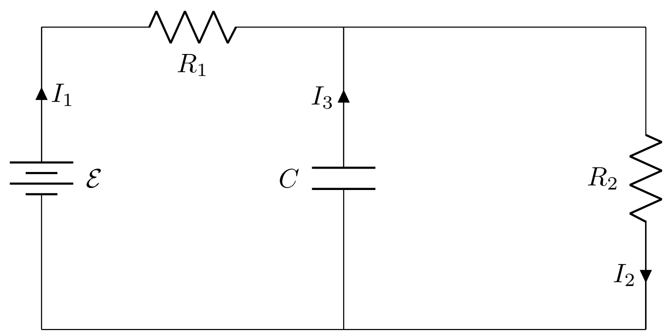

Figure 1: The circuit in the problem, rearranged and with currents labeled

Because of the way the circuit is constructed, the capacitor begins in a fully charged state. We will begin by applying Kirchhoff's junction rule:

$$I_1 + I_3 = I_2 \tag{1}$$

and then the loop rule to the outer loop:

$$0 = \sum V = \mathcal{E} - I_1 R_1 - I_2 R_2 \tag{2}$$

and the right-hand loop:

$$0 = \sum V = q/C - I_2 R_2 \tag{3}$$

where $q(t)$ is the charge on the capacitor at some point in time. From the description of the problem, we also have $R_1 = R_2 = R$ and $I_3 = -dq/dt$ (the minus sign is necessary because the capacitor is discharging: i.e. *losing* charge).

After additional substitutions and rearrangements, this is enough to write the differential equation:

$$RC\frac{dq}{dt} = C\mathcal{E} - 2q \tag{4}$$

which can easily be separated and integrated to find the solution:

$$2q = C\mathcal{E} + Ae^{-2t/RC} \tag{5}$$

With the initial condition $q = C\mathcal{E}$ at $t = 0$ (i.e. the capacitor starts off fully charged), we see that $A = C\mathcal{E}$, and the charge q on the capacitor varies with time according to:

$$q = \frac{C\mathcal{E}}{2}\left(1 + e^{-2t/RC}\right) \tag{6}$$

Referring to Equation (3), it immediately follows that the current I_2 is:

$$\boxed{I_2 = \frac{\mathcal{E}}{2R}\left(1 + e^{-2t/RC}\right)} \tag{7}$$

Additionally, by Equation (2), we get I_1 for free:

$$\boxed{I_1 = \frac{\mathcal{E}}{2R}\left(1 - e^{-2t/RC}\right)} \tag{8}$$

Finally, as the problem requires, we can plot their general shapes, shown in Figure 2.

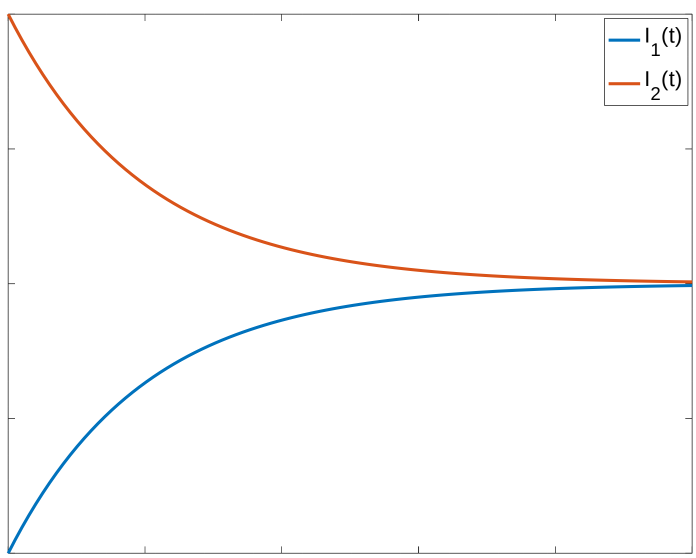

Figure 2: Plot of $I_1(t)$ and $I_2(t)$ against time

Let's see if this result agrees intuitively with the limiting cases $t = 0$ and $t = \infty$. At the beginning, the voltage across C must be equal to $\mathcal{E}$, since it is fully charged with the second switch (not shown in Figure 1) open. By Kirchhoff's law, this implies that no current is flowing in the left-hand loop, so $I_1 = 0$ at the start. On the contrary, since C and R_2 are in parallel, the voltage across them is the same; that is, R_2 starts off with a nonzero voltage drop. A resistor with a nonzero voltage drop must obviously have some current flowing through it, so I_2 starts off at some nonzero value. After a very long time has passed, however, C is once again fully charged for this configuration, so it effectively ceases to conduct in the circuit. Then R_1 and R_2 are in series, and $I_1 = I_2$ after a long time.